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Albert-Ludwigs-Universität Freiburg
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# Model Theory

Sheet 2

Deadline: 30.10.25, 2:30 pm.

**Exercise 1** (14 points)**.**

Consider a language $\mathcal{L} = \{E\}$, where $E$ is a binary relation symbol. Let $T$ be the theory stating that $E$ is interpreted as an equivalence relation with the property that for each $n \geq 1$, there is exactly one equivalence class of size $n$.

a) Provide an axiomatization of $T$ and show that $T$ is consistent.

b) Show with an explicit example that $T$ does not admit quantifier elimination.

c) Let $\mathcal{A} \models T$ be a countable model. Using the Löwenheim-Skolem theorem, show that there exists an elementary extension $\mathcal{A}'$ of $\mathcal{A}$ with cardinality $2^{\aleph_0}$ that has exactly $2^{\aleph_0}$ equivalence classes of cardinality $2^{\aleph_0}$.

Are all models of $T$ with cardinality $2^{\aleph_0}$ isomorphic to $\mathcal{A}'$?

d) Introduce new constant symbols $(c_n)_{1 \leq n \in \mathbb{N}}$ and consider the extension $T_1$ of $T$ in the language $\mathcal{L} \cup \{c_n\}_{1 \leq n \in \mathbb{N}}$ which states that the equivalence class of $c_n$ has exactly $n$ elements. Show that $T_1$ is complete and admits quantifier elimination.

Notice that every model of $T$ can be extended to a model of $T_1$.

e) From Exercise 3 a) of Sheet 1, it follows that $T_1$ is model complete. Is $T$ model complete?

**Hint**: Given $\mathcal{B} \models T$, pick two elements from the equivalence class of size three, etc.

**Exercise 2** (6 points)**.**

Let $T$ be the theory of an infinite set in the empty language $\mathcal{L} = \emptyset$.

a) Given a primitive existential $\mathcal{L}$-formula $\varphi[\bar{x}]$, i.e. of the form $\exists y \psi[\bar{x}, y]$ with $\psi[\bar{x}, y]$ a finite conjunction of atomic and negations of atomic formulae, find an explicit quantifier-free formula $\theta[\bar{x}]$ such that

$$T \vDash \forall \bar{x}(\varphi[\bar{x}] \leftrightarrow \theta[\bar{x}]).$$

Now let $\mathcal{L} = \{<\}$ be the language with a binary relation symbol and $T$ the theory stating that $<$ is interpreted as a dense linear order without endpoints.

b) Show that every primitive existential formula can be explicitly eliminated as in a).

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The exercise sheets can be handed in in pairs. Submit them in the mailbox 3.19 in the basement of the Mathematical Institute.